Name: ______________________

Pid: ______________________

1. (10 points) Check all the correct statements.
   - ◯ The inverse of the permutation $(1, 2, 3)(4, 5)$ is $(2, 1, 3)(4, 5)$.
   - ◯ There are 60 permutations of the cyclic type $(2, 0, 1)$.
   - ◯ Product of the permutations 13245 and 32154 is 23154.
   - ◯ The number of different strings you can get by reordering letters in the word abccc is 30.
   - ◯ If you have 26 balls in 5 boxes, then there is a box with at least 6 balls.

Solution:

1. This permutation is equal to 23154, so the inverse is equal to 31254 and $(2, 1, 3)(4, 5)$ is equal to 31254.
2. The number of permutations of this type is equal to $\frac{(2+3)!}{2!1!1^2 3^1} = 5 \cdot 4 = 20$.
3. The product is equal to 23154.
4. If all the letters are different there are 5! different words, however, we have three "c". Therefore, the answer is $\frac{5!}{3!} = 20$.
5. By the pigeonhole principle, there is a box with $\lceil 26/5 \rceil = 6$ balls.

2. (10 points) Show that if $p(n)$ denotes the number of partitions of the integer n, then

$$\sum_{n\geq 0} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Solution: Note that

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = (1+x+x^2+\dots)(1+x^2+x^3+\dots)\dots(1+x^k+x^{2k}+\dots)\dots$$

Let us determine the coefficient of x^n on the right-hand side. The right-hand side is a sum of products, such that each member comes from a different k. The member from the kth parentheses is of the form x^{ki_k}, and the sum of the exponents of the terms is n. In other words, $1i_1 + 2i_2 + \dots + ki_k + \dots = n$. If we write $\underbrace{1+1+\dots+1}_{i_1 \text{ times}}$ instead of $1i_1$, and in general, $\underbrace{k+k+\dots+k}_{i_k \text{ times}}$ instead of ki_k in the previous equation, we obtain a partition of n into the sum. Using this procedure, each time a product on the right-hand side is equal to x^n, we obtain a partition of n into the sum of parts. Conversely, each partition of n into parts can be associated to a product on the right- hand side, and the statement follows

3. (10 points) Let $f(n)$ be the number of subsets of $[n]$ in which the distance of any two elements is at least three. Find the generating function of $f(n)$.

Solution: Note that if n is part of the subset, then we cannot have $n-1$ or $n-2$ in the subset, so we have $f(n-3)$ ways to choose such a subset. Indeed, we can append n to the end of any good subset of $[n-3]$; if n is not part of our subset, then we obviously have $f(n-1)$ choices. So $f(n) = f(n-1) + f(n-3)$, for all integers $n \geq 3$. Moreover, $f(0) = 1$, $f(1) = 2$, and $f(2) = 3$. Let $F(x)$ be the generating function for $f(n)$. Then we have an equation $F(x) - 3x^2 - 2x - 1 = x(F(x) - 2x - 1) + x^3 F(x)$, from where $F(x) = \frac{1+x+x^2}{1-x-x^3}$.

4. (10 points) Show that any permutation is a product of cycles of length 2 (such cycles are called transpositions).

Solution: It is easy to see that it is enough to prove the statement for the cycles. Consider a permuatation $(i_1, \ldots, i_k)$. It is clear that $(i_1, i_2)(i_2, i_3)$ is equal to (i_1, i_2, i_3) and moreover $(i_1, i_2)(i_2, i_3) \ldots (i_{k-1}, i_k)$ is equal to $(i_1, \ldots, i_k)$. Therefore we proved the statement.