Name: ____________________

Pid: ____________________

1. (10 points) Check all the correct statements.

   ◯ The number of different strings you can get by reordering letters in the word aabbc is 30.

   ◯ There are 25 different strings of length 5 over the alphabet with two letters.

   ◯ If you have 26 balls in 5 boxes, then there is a box with at least 6 balls.

   ◯ There are 6 different surjective functions from [3] to [2].

   ◯ There are 15 variants to put 4 identical balls into 3 different boxes.

Solution:

1. If all the letters are different there are 5! different words, however, we have tow "a" and two "b". Therefore, the answer is $\frac{5!}{2!2!} = 30$.
2. For each letter of the word we have choice of two letters, hence, by the multiplicative law, the answer is $2^5 \neq 25$.
3. By the pigeonhole principle, there is a box with $\lceil 26/5 \rceil = 6$ balls.
4. Note that $S(3,2) = \binom{3}{2} = 3$. Hence, there are $2! \cdot 3 = 6$ surjective functions.
5. There are $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$ ways to put the balls.

2. (10 points) Let us assume that we are given ℓ lines that are not parallel to each other. Prove that there are at least two of them such that angle between them is at most π/ℓ.

Solution: Move all the lines (using parallel shift) such that all of them are going through $(0,0)$. Let us denote angles between lines (in clockwise order) $\alpha_1, \ldots, \alpha_{2\ell}$ respectively and assume that all of them are greater than π/ℓ. In this case we may note that $\sum\limits_{i=1}^{2\ell} \alpha_i > 2\ell \cdot \pi/\ell = 2\pi$, but we know that $\sum\limits_{i=1}^{2\ell} \alpha_i = 2\pi$.

3. (10 points) Prove that for all integers $n > 0$, the sum $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ is at most 2.

Solution: Let us prove a stronger statement:

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

The base case is clear. We prove now the induction step. By the induction hypothesis the following inequality holds

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{(n-1)^2} \leq 2 - \frac{1}{n-1}.$$

Hence,

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n-1} + \frac{1}{n^2} = 2$$

but $\frac{1}{n-1} - \frac{1}{n^2} \geq \frac{1}{n}$. Indeed, $\frac{1}{n-1} \geq \frac{n+1}{n^2}$ is equivalent to $\frac{1}{(n-1)^2} \geq \frac{1}{n^2}$ wcich is true. As a result we proved the induction step.

4. (10 points) Find a closed formula (no summation signs) for the expression $\sum_{i=1}^{n} i^2 \binom{n}{i}(-1)^i$.

Solution: Firstly, Let us consider cases when $n \leq 2$. If $n = 0$, then $\sum_{i=1}^{n} i^2 \binom{n}{i}(-1)^i = 0$. If $n = 1$, then $\sum_{i=1}^{n} i^2 \binom{n}{i}(-1)^i = -1$. If $n = 2$, then $\sum_{i=1}^{n} i^2 \binom{n}{i}(-1)^i = 2$.
Let us now consider other cases. Note that

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Hence, if we derive both sides of the equality we get

$$n \cdot (1+x)^{n-1} = \sum_{i=1}^{n} i \cdot \binom{n}{i} x^{i-1}.$$

Hence, if we substitute $x = -1$ we prove that $\sum_{i=1}^{n} i \cdot \binom{n}{i}(-1)^{i-1} = 0$. Now let us derive the equality once again:

$$n(n-1) \cdot (1+x)^{n-2} = \sum_{i=2}^{n} i(i-1) \cdot \binom{n}{i} x^{i-2}.$$

Using previous argument we prove that $\sum_{i=2}^{n} i(i-1) \cdot \binom{n}{i}(-1)^i = 0$. Since $\sum_{i=0}^{n} i^2 \cdot \binom{n}{i} x^i = x^2 \cdot \sum_{i=2}^{n} i(i-1) \cdot \binom{n}{i} x^{i-2} + x \cdot \sum_{i=1}^{n} i \cdot \binom{n}{i} x^{i-1}$, the answer is 0.

5. (10 points) How many different words one can get by reordering the letters of the word "combinatorics"?

Solution: Note that the word "combinatorics" has 13 letters, two of them are "o", two of them are "c", two of them are "i", and all other are different. Hence, if we assume for a second that all the letters are different, we can get 13! different words. But note that we can exchange "o"s and it does not change the word, we can also exchange "c"s etc. Hence, we need to divide 13! by $2 \cdot 2 \cdot 2$. As a result, the answer is $\frac{13!}{8}$.