

$c(n, k) = \#$ permutations with k cycles.
 $S(n, k) = (-1)^{n-k} c(n, k)$ Stirling numbers of the first kind

Theorem

$$(x)_n = \sum_{k=0}^n \overline{S}(n, k) x^k$$

Theorem

$$x^n = \sum_{k=0}^n \underbrace{S(n, k)}_{\text{Stirling numbers of the second kind}} (x)_k$$

$$(x)_k = x(x-1)\dots(x-k+1)$$

Theorem

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k)$$

$$(123)(45) = (231)(45) = (45)(123) \dots$$

Definition

We say that a canonical cycle form of perm. is the form where all the cycles start with a largest number

And cycles are ordered in the increasing order of the first elements

$(312)(54)$ is the can. form of

23154

Exercise

write the perm. in the can. cyc. form

$(2\ 4\ 3\ 1)$ $(1\ 5)$ (6) (7)

$\begin{array}{cccc} | & | & | & | \\ (4\ 3\ 2) & (5\ 1) & (6) & (7) \end{array} \leftarrow \text{Answer}$

Note that if I erase the par.

i.g. consider $(4\ 3\ 2\ 1\ 5\ 6\ 7)$

The can. form of $(2\ 5\ 3)$ $(1\ 4)$ (6) (7) is

$\begin{array}{cccc} | & | & | & | \\ (5\ 3\ 2) & (4\ 1) & (6) & (7) \end{array}$

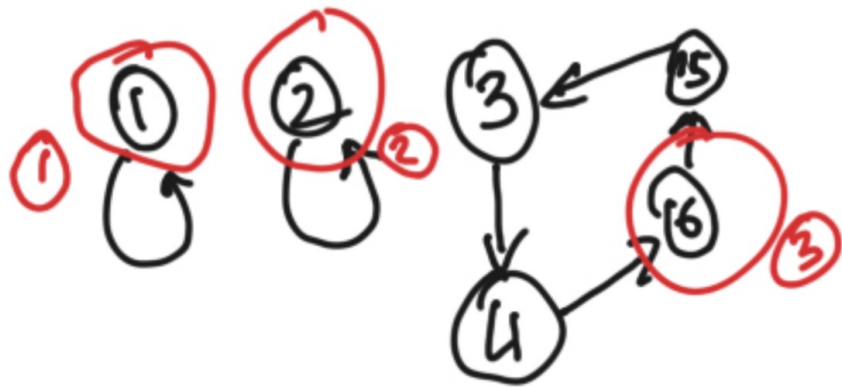
$(4\ 1)$ $(5\ 3\ 2)$ (6) $(7) \leftarrow \text{Answer}$

Consider $G: S_n \rightarrow S_n$ such that

$\pi \rightarrow$ write it in can. cycle form \rightarrow erase par. \rightarrow
 \rightarrow interpret as a perm in the one-line not \rightarrow
 $\rightarrow G(\pi)$

Example

$$124635 = (1)(2)(6534) \xrightarrow{G} 126534$$



Theorem

G is a bijection

Problem

Given k numbers $a_1, \dots, a_k \in [n]$.

How many permutations have a_1, \dots, a_k in the same cycle?

Answer: $\frac{n!}{k}$

WLOG $a_1 = n$

So we are interested in the cases when a_2, \dots, a_k are to the right of n