Name: ______________________

Pid: ______________________

1. (10 points) Let $S \subseteq \mathbb{N}$ be a nonempty set. Show that S is decidable iff there is a funciton $f : \mathbb{N} \to \mathbb{N}$ such that f is computable, f is nondecreasing, and $\mathrm{Im}\ f = S$.

Solution: Let us assume that S is decidable and $\mathcal{A}$ decides S. Let $x_{\min}$ be the minimal element of S. Consider the following algorithm.

function $\mathcal{F}(n)$
    if $n < x_{\min}$ **then**
        return $x_{\min}$
    end if
    Let $x \leftarrow n$
    while $\neg\mathcal{A}(x)$ and $x > x_{\min}$ **do**
        $x \leftarrow x - 1$
    end while
    return x
end function

It is clear that this algorithm computes the total function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(x) = \begin{cases} x_{\min} & \text{if } x < x_{\min} \\ \max\{y \in S \ : \ y \leq x\} & \text{otherwise} \end{cases}.$$

Therefore f is nondecreasing. We need to prove now that $\mathrm{Im}\ f = S$. To prove this firt we note that it is easy to see that $\mathrm{Im}\ f \subseteq S$. In addtion, if $x \in S$, then $f(x) = x$, which implies that $\mathrm{Im}\ f = S$.

Let us now prove the statement in the opposite dirrection. Assume that there is a total nondecreasing funciton $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{Im}\ f = S$.

- If S is finite, then we prove in class that it is decidable.
- Let S be an infinite set, and let $\mathcal{F}$ be an algorithm computing f. Consider the following algorithm

  1: **function** $\mathcal{A}(x)$
  2:     **while** $\mathcal{F}(n) < x$ **do**
  3:         $n \leftarrow n + 1$
  4:     **end while**
  5:     **if** $\mathcal{F}(n) = x$ **then**
  6:         **return** 1
  7:     **else**
  8:         **return** 0
  9:     **end if**
  10: **end function**

  In lines 2-4 this algorithm finds the minimal n such that $f(n) \geq x$ (it always exists since S is infinite). If $x \in S$, then such a minumal n is equal x, and the algorithm returns 1. Otherwise $f(n) \neq x$ and the algorithm returns 0.

2. (10 points) Let $A, B \subseteq \mathbb{N}$ be enumeratable sets. Show that $A \times B$ is enumeratable.

Solution: Let $\mathcal{A}$ and $\mathcal{B}$ be semideciding algorithms for A and B, respectively. Consider the following algorithm.

function $\mathcal{C}(x, y)$
  $\mathcal{A}(x)$
  $\mathcal{B}(x)$
  return 1
end function

Note that if $x \in A$ and $y \in B$, then the algorithm return 1. Otherwise the algorithm never terminates. Therefore, if $(x, y) \in A \times B$, then the algorithm return 1. Otherwise the algorithm never terminates.