Name: ______________________

Pid: ______________________

1. (10 points) Show that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$.

**Solution:** We prove the statement using induction by $n$. The base case for $n = 1$ is true since $1^2 = 1$ and $\frac{1(1+1)(2+1)}{6} = 1$.

Now we need to prove the induction step from $n$ to $n+1$. Assume that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. By the hypothesis, $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$. Note that

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6n^2 + 12n + 6}{6} =$$
$$\frac{n^3 + 2n^2 + n^2 + n + 6n^2 + 12n + 6}{6} = \frac{n^3 + 9n^2 + 13n + 6}{6} =$$
$$\frac{(n+1)(n+2)(2n+3)}{6}.$$

Hence, the induction step is true and by the induction principle, $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers $n$.

2. (10 points) Let $a_0 = 2$, $a_1 = 5$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$. Show that $a_n = 3^n + 2^n$ for all integers $n \geq 0$.

**Solution:** We prove the statement using induction by $n$ for $n \geq 0$. The base cases for $n = 0$ and $n = 1$ are true since $3^0 + 2^0 = 1 + 1 = 2 = a_0$ and $3^1 + 2^1 = 3 + 2 = 5 = a_1$.

Let us now prove the induction step from $n$ and $n-1$ to $n+1$. Assume that $a_{n-1} = 3^{n-1} + 2^{n-1}$ and $a_n = 3^n + 2^n$. Note that $a_{n+1} = 5a_n - 6a_{n-1}$; hence, by the induction hypothesis, $a_{n+1} = 5(3^n + 2^n) - 6(3^{n-1} + 2^{n-1}) = (5 \cdot 3 - 6)3^{n-1} + (5 \cdot 2 - 6)2^{n-1} = 9 \cdot 3^{n-1} + 4 \cdot 2^{n-1} = 3^{n+1} + 2^{n+1}$.

As a result, the induction step is true and by the induction principle, $a_n = 3^n + 2^n$ for all integers $n \geq 0$. (Note that we proved a stronger statement than it was asked in the problem.)

3. (10 points) Let $n$ be a positive integer and $A_1, \ldots, A_n$ be some sets. Let us define union of these sets as follows:

   1. $\cap_{i=1}^{1} A_i = A_1$,
   2. $\cap_{i=1}^{k+1} A_i = (\cap_{i=1}^{k} A_i) \cap A_{k+1}$.

   Show that $\cap_{i=1}^{n} \{x \in \mathbb{N} \,:\, i \leq x \leq n\} = \{n\}$.

**Solution:** We prove using induction by $m$ that $\cap_{i=1}^{m} \{x \in \mathbb{N} \,:\, i \leq x \leq n\} = \{m, m+1, \ldots, n\}$. The base case is for $m = 1$ is true since

$$\cap_{i=1}^{1} \{x \in \mathbb{N} \,:\, i \leq x \leq n\} = [n] = \{1, 2, \ldots, n\}.$$

Let us now prove the induction step from $m$ to $m+1$. Assume that $\cap_{i=1}^{m} \{x \in \mathbb{N} \,:\, i \leq x \leq n\} = \{m, m+1, \ldots, n\}$. Note that

$$\cap_{i=1}^{m+1} \{x \in \mathbb{N} \,:\, i \leq x \leq n\} = \big( \cap_{i=1}^{m} \{x \in \mathbb{N} \,:\, i \leq x \leq n\} \big) \cap \{x \in \mathbb{N} \,:\, m+1 \leq x \leq n\}.$$

Therefore

$$\cap_{i=1}^{m+1} \{x \in \mathbb{N} \,:\, i \leq x \leq n\} = \{m, m+1, \ldots, n\} \cap \{m+1, \ldots, n\} = \{m+1, \ldots, n\}.$$

Hence, by the induction principle, the statement is true for all $m$. As a result, we proved for $m = n$ that $\cap_{i=1}^{n} \{x \in \mathbb{N} \,:\, i \leq x \leq n\} = \{n\}$.

4. (10 points) Let $U$ be the set of sequences of the following symbols: "+", "·", "$x_1$", …, "$x_n$". Let $B = \{x_i \ : \ i \in [n]\}$; i.e., $B$ is the set of sequences consisting of only one symbol $x_i$. Let $\mathcal{F} = \{f_+, f_\cdot\}$, where $f_+(F_1, F_2) = (F_1 + F_2)$ and $f_\cdot(F_1, F_2) = (F_1 \cdot F_2)$ (by $(F_1 \# F_2)$ we denote the sequence obtained by concatenating "(", $F_1$, "#", $F_2$, and ")"). Let $S$ be the set generated by $\mathcal{F}$ from $B$.

For $s : [n] \to \{0, 1\}$ and $F \in S$, we define the function $\mathrm{val}(F, s)$ using structural recursion as follows.

1. $\mathrm{val}(x_i, s) = s(i)$,
2. $\mathrm{val}((F_1 + F_2), s) = \mathrm{val}(F_1, s) + \mathrm{val}(F_2, s)$,
3. $\mathrm{val}((F_1 \cdot F_2), s) = \mathrm{val}(F_1, s) \cdot \mathrm{val}(F_2, s)$.

Let $F_1, \dots, F_n \in S$. Let us define the sum of these formulas as follows:

1. $\sum_{i=j}^{j} F_i = F_j$,
2. $\sum_{i=j}^{j+k} F_i = f_+(\sum_{i=j}^{j+k-1} F_i, F_{j+k})$ for $k \geq 1$.

Show that $\mathrm{val}(\sum_{i=1}^{n} x_i, s) = \mathrm{val}(\sum_{i=1}^{n} x_{n-i+1}, s)$ for any $s$.

**Solution:** Before we start working with the arithmetic formulas, let us prove several statements for real number. Let $a_1, \dots, a_n$ be some real numbers. We show that $\sum_{i=m}^{m+n} a_i = a_m + \sum_{i=m+1}^{m+n} a_i$ for $n \geq 1$ using induction by $n$. The base case is true for $n = 1$ since $\sum_{i=m}^{m+1} a_i = a_m + a_{m+1} = a_m + \sum_{i=m+1}^{m+1} a_i$.

Let us now prove the induction step from $n$ to $n+1$. Assume that $\sum_{i=m}^{m+n} a_i = a_m + \sum_{i=m+1}^{m+n} a_i$. Note that by the induction hypothesis,

$$\sum_{i=m}^{m+n+1} a_i = \sum_{i=m}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n+1}.$$

Using this statement we may show that $\sum_{i=1}^{m} a_i = \sum_{i=n-m+1}^{n} a_{n-i+1}$ for $m \geq 1$ using induction by $m$. The base case is true since $\sum_{i=1}^{1} a_i = a_1 = \sum_{i=n}^{n} a_{n-i+1}$. To prove the induction step from $m$ to $m+1$; assume $\sum_{i=1}^{m} a_i = \sum_{i=n-m+1}^{n} a_{n-i+1}$. Note that the hypothesis implies that

$$\sum_{i=1}^{m+1} a_i = \sum_{i=1}^{m} a_i + a_{m+1} = \sum_{i=n-m+1}^{n} a_{n-i+1} + a_{m+1} = \sum_{i=n-m}^{n} a_{n-i+1}.$$

Therefore by the induction hypothesis, $\sum_{i=1}^{m} a_i = \sum_{i=n-m+1}^{n} a_{n-i+1}$ for $m \geq 1$. If we consider $m = n$, we get $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a_{n-i+1}$.

Let us now explain how to get this statement for arithmetic formulas. Let $F_1, \dots, F_m$ be some arithmetic formulas. Then we may show that $\mathrm{val}(\sum_{i=1}^{m} F_i, s) = \sum_{i=1}^{m} \mathrm{val}(F_i, s)$ for all $s$. Fix some $s$; we prove this statement also using induction. The base case for $m = 1$ is true since $\sum_{i=1}^{1} F_i = F_1$ and $\sum_{i=1}^{1} \mathrm{val}(F_i, s) = \mathrm{val}(F_1, s)$. To prove the induction step from $m$ to $m+1$; assume $\mathrm{val}(\sum_{i=1}^{m} F_i, s) = \sum_{i=1}^{m} \mathrm{val}(F_i, s)$. Note that $\sum_{i=1}^{m+1} F_i = f_+(\sum_{i=1}^{m} F_i, F_{m+1})$, and $\mathrm{val}(f_+(\sum_{i=1}^{m} F_i, F_{m+1}), s) = \mathrm{val}(\sum_{i=1}^{m} F_i, s) + \mathrm{val}(F_{m+1}, s)$. Hence,

$$\mathrm{val}(\sum_{i=1}^{m+1} F_i, s) = \mathrm{val}(\sum_{i=1}^{m} F_i, s) + \mathrm{val}(F_{m+1}, s) = \sum_{i=1}^{m} \mathrm{val}(F_i, s) + \mathrm{val}(F_{m+1}, s) = \sum_{i=1}^{m+1} \mathrm{val}(F_i, s).$$

Using all these statement, we may notice that

$$\mathrm{val}(\sum_{i=1}^{n} x_i, s) = \sum_{i=1}^{n} \mathrm{val}(x_i, s) = \sum_{i=1}^{n} \mathrm{val}(x_{n-i+1}, s) = \mathrm{val}(\sum_{i=1}^{n} x_{n-i+1}, s).$$