Name: ______________________

Pid: ______________________

1. (10 points) Let us formulate the pigeonhole principle using propositional formulas. $\Omega = \{x_{1,1}, \ldots, x_{n+1,1}, x_{1,2} \ldots, x_{n+1,n}\}$ (informally $x_{i,j}$ is true iff the ith pigeon is in the jth hole). Consider the following propositional formulas on the variables from Ω.

   - L_i ($i \in [n+1]$) is equal to $\bigvee_{j=1}^{n} x_{i,j}$. (Informally this formula says that the ith pigeon is in a hole.)
   - R_j ($j \in [n]$) is equal to $\bigvee_{i_1=1}^{n+1} \bigvee_{i_2=i_1+1}^{n+1} (x_{i_1,j} \wedge x_{i_2,j})$. (Informally this formula says that there are two pigeons in the jth hole.)

   Show that there is a natural deduction proof of $\left(\bigwedge_{i=1}^{n+1} L_i\right) \implies (\bigvee_{i=1}^{n} R_i)$.

2. (10 points) Let $\phi = \bigvee_{i=1}^{m} \lambda_i$ be a clause; we say that the width of the clause is equal to m. Let $\phi = \bigwedge_{i=1}^{\ell} \chi_i$ be a formula in CNF; we say that the width of ϕ is equal to the maximal width of χ_i for $i \in [\ell]$.

   Let $p_n : \{T, F\}^n \to \{T, F\}$ such that $p_n(x_1, \ldots, x_n) = T$ iff the set $\{i \ : \ x_i = T\}$ has odd number of elements. Show that any CNF representation of p_n has width at least n.

3. (10 points) Write a natural deduction derivation of $(W \vee Y) \implies (X \vee Z)$ from hypotheses $W \implies X$ and $Y \implies Z$.

4. (10 points) We say that a clause C can be obtained from clauses A and B using the *resolution* rule if $C = A' \vee B'$, $A = x \vee A'$, and $B = \neg x \vee B'$, for some variable x.

   We say that a clause C can be derived from clauses $A_1, \ldots, A_m$ using resolutions if there is a sequence of clauses $D_1, \ldots, D_\ell = C$ such that each D_i

   - is either obtained from clauses D_j and D_k for $j, k < i$ using the *resolution* rule, or
   - is equal to A_j for some $j \in [m]$, or
   - is equal to $D_j \vee E$ for some $j < i$ and a clause E.

   Show that if $A_1, \ldots, A_m$ semantically imply C, then C can be derived from clauses $A_1, \ldots, A_m$ using resolutions.