

Name: \_\_\_\_\_

Pid: \_\_\_\_\_

1. (10 points) Give a natural deduction derivation of  $\exists x (A(x) \vee B(x))$  from  $\exists x A(x) \vee \exists x B(x)$ .

**Solution:**

1	$\exists x A(x) \vee \exists x B(x)$	
2	$\exists x A(x)$	
3	$A(y)$	
4	$A(y) \vee B(y)$	$\vee I, 3$
5	$\exists x (A(x) \vee B(x))$	$\exists I, 4$
6	$\exists x (A(x) \vee B(x))$	$\exists E, 2, 3-5$
7	$\exists x B(x)$	
8	$B(y)$	
9	$A(y) \vee B(y)$	$\vee I, 8$
10	$\exists x (A(x) \vee B(x))$	$\exists I, 9$
11	$\exists x (A(x) \vee B(x))$	$\exists E, 7, 8-10$
12	$(\exists x (A(x) \vee B(x)))$	$\vee E, 1, 2-5, 6-9$

2. (10 points) Let us consider the following formulas on the variables from the set  $\{x_0, \dots, x_n\}$ .
1. The formula  $I_n$  is equal to  $x_0$ .
  2. The formula  $S_{n,i}$  is equal to  $x_{i-1} \implies x_i$ .
  3. The formula  $T_n$  is equal to  $x_n$ .

Show that there is a natural deduction derivation of  $T_n$  from  $I_n \wedge \bigwedge_{i=1}^n S_{n,i}$ .

**Solution:** By completeness theorem, it is enough to show that  $I_n \wedge \bigwedge_{i=1}^n S_{n,i} \models T_n$ .

Consider an assignement  $\rho$  such that  $(I_n \wedge \bigwedge_{i=1}^n S_{n,i})|_\rho = T$ . Note that this implies that  $\rho(x_0)$  is true, and  $\rho(x_{i-1}) \implies \rho(x_i)$  is true for all  $i \in [n]$ . We prove using induction that for any  $k$ ,  $\rho(x_k)$  is true. The base case is clear since  $\rho(x_0)$  is true. Let us now prove the induction step. Assume that  $\rho(x_k)$  is true. So  $x_{k+1}$  is true since  $\rho(x_k) \implies \rho(x_{k+1})$  is true.

As a result we proved that  $\rho(x_n)$  is true, which implies that  $T|_\rho$  is true.

3. (10 points) Let  $\phi = \bigvee_{i=1}^m \lambda_i$  be a clause; we say that the width of the clause is equal to  $m$ . Let  $\phi = \bigwedge_{i=1}^{\ell} \chi_i$  be a formula in CNF; we say that the width of  $\phi$  is equal to the maximal width of  $\chi_i$  for  $i \in [\ell]$ .

Let  $m_n : \{T, F\}^n \rightarrow \{T, F\}$  such that  $m_n(x_1, \dots, x_n) = T$  iff the number of elements in the set  $\{i : x_i = T\}$  is divisible by 3.

Show that any CNF representation of  $m_n$  has width at least  $n - 2$ .

**Solution:** Assume that there is a CNF  $\phi$  of width at most  $n - 3$  representing  $m_n$ . Let  $C = x_{i_1}^{u_1} \vee \dots \vee x_{i_\ell}^{u_\ell}$  be some clause of  $\phi$ . Let us consider the variables  $y_1$  and  $y_2$  be the variables not equal to  $x_{i_1}, \dots, x_{i_\ell}$ . We define  $\rho_{v_1, v_2}$  such that

$$\rho_{v_1, v_2}(x) = \begin{cases} \neg u_j & \text{if } x = x_{i_j} \text{ for } j \in [\ell] \\ v_j & \text{if } x = y_j \text{ for } j \in [2] \\ F & \text{otherwise} \end{cases}$$

Note that  $C|_{\rho_{v_1, v_2}}$  is false and as a result,  $\phi|_{\rho_{v_1, v_2}}$  is also false. Let  $N = |\{j : u_j = F\}|$ . We need to consider the following cases.

- If  $N \equiv 0 \pmod{3}$ , then  $m_n(\rho_{F, F}(x_1), \dots, \rho_{F, F}(x_n)) = T$  which contradicts to  $\phi|_{\rho_{v_1, v_2}} = F$ .
- If  $N \equiv 1 \pmod{3}$ , then  $m_n(\rho_{T, T}(x_1), \dots, \rho_{T, T}(x_n)) = T$  which contradicts to  $\phi|_{\rho_{v_1, v_2}} = F$ .
- If  $N \equiv 2 \pmod{3}$ , then  $m_n(\rho_{T, F}(x_1), \dots, \rho_{T, F}(x_n)) = T$  which contradicts to  $\phi|_{\rho_{v_1, v_2}} = F$ .

4. (10 points) Let  $A\Delta B = (A \cup B) \setminus (A \cap B)$ ; we say that  $A\Delta B$  is the symmetric difference of  $A$  and  $B$ . Let  $\Omega$ , and  $A_1, \dots, A_n \subseteq \Omega$  be some sets. We say that  $\Delta_{i=1}^1 A_i = A_1$  and  $\Delta_{i=1}^{k+1} A_i = (\Delta_{i=1}^k A_i)\Delta A_{k+1}$ . Show that

$$\Delta_{i=1}^n A_i = \{x \in \Omega : x \in A_i \text{ for odd number of } i \in [n]\}.$$

**Solution:** We prove the statement using induction by  $n$ . The base case for  $n = 1$  is obvious. Let us prove the induction step. Assume that

$$\Delta_{i=1}^k A_i = \{x \in \Omega : x \in A_i \text{ for odd number of } i \in [k]\}.$$

Note that

$$\Delta_{i=1}^{k+1} A_i = (\{x \in \Omega : x \in A_i \text{ for odd number of } i \in [k]\})\Delta A_{k+1}.$$

Let us fix some  $x \in \Omega$  and consider the following cases.

- If  $x \in \{x \in \Omega : x \in A_i \text{ for odd number of } i \in [k]\}$  and  $x \in A_{k+1}$ , then  $x \notin \Delta_{i=1}^{k+1} A_i$  and for an even number of  $i \in [k+1]$ ,  $x \in A_i$ .
- If  $x \notin \{x \in \Omega : x \in A_i \text{ for odd number of } i \in [k]\}$  and  $x \in A_{k+1}$ , then  $x \in \Delta_{i=1}^{k+1} A_i$  and for an odd number of  $i \in [k+1]$ ,  $x \in A_i$ .
- If  $x \in \{x \in \Omega : x \in A_i \text{ for odd number of } i \in [k]\}$  and  $x \notin A_{k+1}$ , then  $x \in \Delta_{i=1}^{k+1} A_i$  and for an odd number of  $i \in [k+1]$ ,  $x \in A_i$ .
- If  $x \notin \{x \in \Omega : x \in A_i \text{ for odd number of } i \in [k]\}$  and  $x \notin A_{k+1}$ , then  $x \notin \Delta_{i=1}^{k+1} A_i$  and for an even number of  $i \in [k+1]$ ,  $x \in A_i$ .

5. (10 points) Let  $\mathcal{S}$  be a signature with two predicate symbols  $=$  and  $S$  such that the first is binary and the last is ternary.

Let us consider the structure  $\mathfrak{M}$  such that it corresponds to the points on a two-dimensional plane,  $=$  is a standard equality, and  $S(x, y, z)$  is true iff  $|xz| = |yz|$ .

Let  $R$  be a relation such that  $(A, B, C) \in R$  iff  $A$ ,  $B$ , and  $C$  lay on the same line. Show that  $R$  is representable in  $\mathfrak{M}$ .

**Solution:** Let us fix some points  $A$ ,  $B$ , and  $C$ . Note that they lay on a line iff  $C$  is the only point such that the distance between  $A$  and  $C$  is  $|AC|$  and the distance between  $B$  and  $C$  is  $|BC|$ . Hence, the following formula represents  $R$ :  $\forall C' (S(C, C', A) \wedge S(C, C', B)) \implies C' = C$ .

6. (10 points) Let us define the set  $S$  defined as follows:

- $3 \in S$  and
- if  $x \in S$  and  $y \in S$ , then  $(x + y) \in S$ .

Show that  $S = \{3k : k \in \mathbb{N}\}$ .

**Solution:** The statement consists of two parts:

- $S \subseteq \{3k : k \in \mathbb{N}\}$  and
- $S \supseteq \{3k : k \in \mathbb{N}\}$ .

We prove the first one using structural induction.

**(base case)** Note that  $3 \in \{3k : k \in \mathbb{N}\}$ .

**(induction step)** Let  $z = x + y$  and  $\{3k : k \in \mathbb{N}\}$ . It is clear that  $z \in \{3k : k \in \mathbb{N}\}$ .

Hence, by the structural induction theorem,  $S \subseteq \{3k : k \in \mathbb{N}\}$ .

The second statement can be proved using induction by  $k$ . The base case for  $k = 1$  is true since  $3 \in S$ . Let us now prove the induction step. By the induction hypothesis,  $3k \in S$ ; hence,  $3k + 3 = 3(k + 1) \in S$ .

7. (10 points) Let  $f, g_1, \dots, g_n : \mathbb{R}^\ell \rightarrow \mathbb{R}$ . We say that the equation  $f(x) = 0$  can be derived from the equations  $g_1(x) = 0, \dots, g_n(x) = 0$  iff there is a sequence of functions  $h_1, \dots, h_m : \mathbb{R}^\ell \rightarrow \mathbb{R}$  such that  $h_m = f$  and for each  $i \in [m]$ ,

- either  $h_i$  is equal to  $g_j$  for some  $j \in [n]$ , or
- $h_i = h_j + h_k$  for some  $1 \leq j, k < i$ , or
- $h_i = c \cdot h_j$  for some  $1 \leq j < i$  and some  $c \in \mathbb{R}$ .

Show that if the equation  $f(x) = 0$  can be derived from the equations  $g_1(x) = 0, \dots, g_n(x) = 0$ , then for any  $v \in \mathbb{R}^\ell$ ,  $f(v) = 0$  provided that  $g_1(v) = \dots = g_n(v) = 0$ .

**Solution:** Let us fix some  $f, g_1, \dots, g_n : \mathbb{R}^\ell \rightarrow \mathbb{R}$ . Let  $h_1, \dots, h_m : \mathbb{R}^\ell \rightarrow \mathbb{R}$  be the derivation of the equation  $f(x) = 0$  from the equations  $g_1(x) = 0, \dots, g_n(x) = 0$ . Consider  $v$  such that  $g_1(v) = \dots = g_n(v) = 0$ .

We prove using induction by  $k$  that  $h_k(v) = 0$ . The base case for  $k = 1$  is clear since  $h_1$  should be equal to  $g_j$  for some  $j \in [n]$  so  $h_1(v) = 0$ . Let us now prove the induction step from  $1, \dots, k-1$  to  $k$ . By the induction hypothesis,  $h_1(v) = \dots = h_{k-1}(v) = 0$ .

- If  $h_k = g_j$  for some  $j \in [n]$ , then  $h_k(v) = 0$  since  $g_j(v) = 0$ .
- If  $h_k = h_i + h_j$  for some  $i, j < k$ , then  $h_k(v) = 0$  since  $h_i(v) = h_j(v) = 0$ .
- If  $h_k = c \cdot h_i$  for some  $i < k$ , then  $h_k(v) = 0$  since  $c \cdot h_i(v) = c \cdot 0$ .