Lecture 7: Relations

Definition

R is a k-ary relation on $X_1 \dots X_n$ iff $R \subseteq X_1 \times \dots \times X_k$.

We say that $(x_1 \dots x_k) \in X_1 \times \dots \times X_k$ is in relation R iff $(x_1 \dots x_k) \in R$

If $k=2$, we say that R is a binary relation and instead of writing $(x_1, x_2) \in R$ we write $x_1 R x_2$

If $X_1 = X_2 = \dots = X_k = X$, then R is a relation on X.

Definition

Let R be a binary relation on a set X.
We say that R is an equivalence relation iff the following is true.

(reflexivity) for any $x \in X$, xRx.

(symmetry) for any $x, y \in X$, xRy iff yRx

(transitivity) For any $x, y, z \in X$, if xRy and yRz, then xRz.

Example

1. The relation "have the same cardinality" is an equivalence relation.

2. Let $n \in \mathbb{N}$. We say that $x, y \in \mathbb{Z}$ are equivalent modulo n (we write it as $x \equiv y \pmod{n}$) iff $x-y$ is divisible by n.

$$x - y = nk \qquad y - z = nl \Rightarrow \underbrace{x - y + y - z}_{x - z} = n(k+l)$$

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3. Let S be a set of symbols of the form $\frac{x}{y}$, where $x, y \in \mathbb{Z}$ and $y \neq 0$.

Consider a relation $\sim$ on S s.t.

$$\frac{a}{b} \sim \frac{c}{d} \quad \text{iff} \quad ad = bc$$

Exercise Show that $\sim$ is an eq. rel.

1. $\frac{a}{b} \sim \frac{a}{b} \iff ab = ab$ for any $a, b \in \mathbb{Z}, b \neq 0$

2. $\frac{a}{b} \sim \frac{c}{d} \overset{?}{\iff} \frac{c}{d} \sim \frac{a}{b}$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$ad = bc \iff cb = ad$

3. assume

$\frac{a}{b} \sim \frac{c}{d} \quad \frac{c}{d} \sim \frac{e}{f} \overset{?}{\Longrightarrow} \frac{a}{b} \sim \frac{e}{f}$

$$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow$$

$ad = bc \quad cf = de \quad af = eb$

$\Updownarrow$

$a = \frac{bc}{d} \iff af = \frac{bc}{d} f \Big\} \Longrightarrow af = \frac{bde}{d} \iff af = be$