Lecture 2 Sets

Definition

A set is a well-defined collection of objects

Exercise

- Give four collections that are sets
- Give two collections that aren't sets

Definition Let S be a set and x be some object. We denote the statement „x belongs to S" by

$$x \in S$$

$$\in$$

Exercise Which statements are true?

- $-1 \in \mathbb{N}$
- $\sqrt{2} \in \mathbb{Q}$
- $\sqrt{2} \in \mathbb{R}$
- $-1 \in \mathbb{Z}$

Definition

We use several ways to define a set

Listing elements: $\{1,2,3\}$ $\{1,2,3\ldots\}$

Conditional definitions: $[n] = \{x \in \mathbb{Z} : 1 \le x \le n\}$

Constructive definitions:

$$\{x^2 : x \in \mathbb{Z}\} = \{x \in \mathbb{R} : x^2 \in \mathbb{Z}\}$$

Exercise Show that in any vector space V, if a set of vectors $\{x_1, \dots, x_n\}$ spans V, then there is a basis $B \subseteq \{x_1, \dots, x_n\}$ of V.

$\downarrow$ they are essentially the same

We prove using induction by n that

For any set $S \subseteq V$ of size n, if S spans V, then there is a basis $B \subseteq S$ of V

Now let's prove for $n=1$. In this case $S = \{x\}$ and is L.I.. Therefore S is a basis.

Assume that the statement is true for k.

Consider S consisting of $k+1$ vectors
$$S = \{x_1, \dots, x_{k+1}\}$$

- If S is L.I., then $B = S$ is a basis.

- Otherwise, WLOG $x_{k+1} = \alpha_1 x_1 + \dots + \alpha_k x_k$

  Therefore $\{x_1 \dots x_k\}$ spans V. By the I.H., there is $B \subseteq \{x_1 \dots x_k\} \subseteq S$ s.t. B is a basis.