Lecture 16

We proved that if $\psi|_\beta = T$ for any β, then there is a derivation of ψ.

Theorem (Completeness)
Let $\varphi_1 \dots \varphi_n, \psi$ be some formulas on Ω.
Then if $\varphi_1 \dots \varphi_n \models \psi$, then we can derive ψ from $\varphi_1 \dots \varphi_n$

Assume $\varphi_1 \models \psi$. What can we say about $(\varphi_1 \Rightarrow \psi)$?

Therefore if $\varphi_1 \models \psi$, there is a derivation of $(\varphi_1 \Rightarrow \psi)$

φ_1

$\vdots$

$\varphi_1 \Rightarrow \psi$

ψ by elim. of $\Rightarrow$

If $\varphi_1 \ldots \varphi_n \models \psi$, then $(\varphi_1 \wedge \varphi_2 \ldots \wedge \varphi_n) \Rightarrow \psi$ is a tautology and there is a derivation of $(\varphi_1 \wedge \varphi_2 \ldots \wedge \varphi_n) \Rightarrow \psi$.

φ_1

$\vdots$

φ_n

$\vdots$

$(\varphi_1 \wedge \dots \wedge \varphi_n) \Rightarrow \psi$

$\varphi_1 \wedge \varphi_2$

$\vdots$

$\varphi_1 \wedge \varphi_2 \wedge \dots \varphi_n$ by introd. of $\wedge$

ψ by elim of $\Rightarrow$

What if $\varphi \neq \psi$ can we derive ψ from φ.

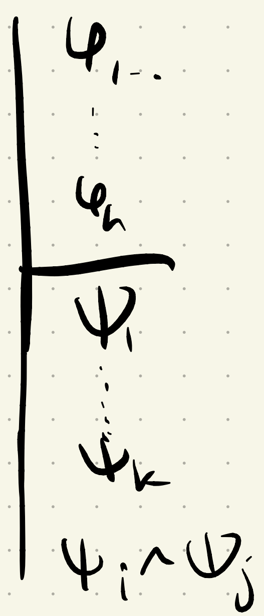
Or what if ψ is not a tautology, can we derive ψ?

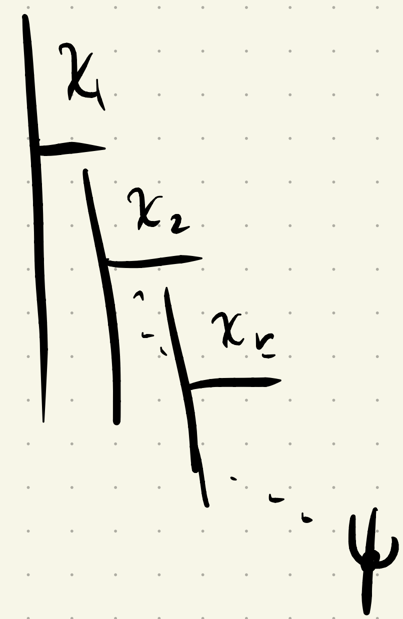
Theorem (Soundness)

Let $\varphi_1, \dots \varphi_n, \psi$ be prop. formulas on Ω.

If there is a der. of ψ from $\varphi_1, \dots \varphi_n$, then $\varphi_1, \dots \varphi_n \models \psi$.

1





2

φ_1

$\vdots$

φ_k

$\psi_1 \wedge \psi_2$

ψ_1

3

φ_1

$\vdots$

φ_k

ψ_1

$\psi_1 \vee \psi_2$

4

φ_1

$\vdots$

φ_k

$A \vee B \leftarrow \varphi_1 \ldots \varphi_k \models A \vee B$

A

$C \leftarrow \varphi_1 \ldots \varphi_k, A \models C$

B

$C \leftarrow \varphi_1 \ldots \varphi_k, B \models C$

$C \leftarrow \varphi_1 \ldots \varphi_k \models C$

φ_1

$\vdots$

φ_k

$A \leftarrow \varphi_1 \ldots \varphi_k \models A$

$\neg A \leftarrow \varphi_1 \ldots \varphi_k \models \neg A$

$\bot \leftarrow \varphi_1 \ldots \varphi_k \models \bot$

$\Omega = \{ x_{ij} \mid i,j \in \mathbb{N} \}$ if All the formulas

- x_{1i} for all $i \in \mathbb{N}$
- $(x_{ij} \wedge x_{jk}) \Rightarrow x_{ik}$ for all $i \in \mathbb{N}$
- $\neg(x_{ij} \wedge x_{ji})$ - for $i \neq j \in \mathbb{N}$
- x_{ii} for $i \in \mathbb{N}$
- $x_{i\,i+1}$ for $i \in \mathbb{N}$

are true

by induction

Then x_{ij} is true for $i < j$

Theorem (compactness theorem)

Let $\psi, \varphi_1, \ldots, \varphi_k, \ldots$ be some prop. formulas.

If $\varphi_1, \ldots, \varphi_k, \ldots \models \psi$, then there are $i_1, \ldots, i_\ell$

s.t. $\varphi_{i_1}, \ldots, \varphi_{i_\ell} \models \psi$