Lecture 10: Propositional formulas

Let Ω be a set of variables

Then

- $x \in \Omega$ is a prop. formula.
- $(\varphi_1 \wedge \varphi_2)$, $(\varphi_1 \vee \varphi_2)$, $\neg \varphi_1$, $(\varphi_1 \Rightarrow \varphi_2)$ are p. formulas provided that φ_1, φ_2 are p. formulas.

Definition

We say that ρ is an assignement to the variables from Ω if $\rho : \Omega \to \{T, F\}$
$\phantom{\rho : \Omega \to \{} 0 \quad 1$

Let $\Omega = \{x_1 \ldots x_n\}$

An assignement that assignes v_i to x_i is also denoted as $x_1 = v_1, \ldots, x_n = v_n$

Definition

Let Ω be a set of variables, and ρ be an assignement to Ω.

The value $\varphi|_\rho$ of a prop. formula φ on Ω with respect to ρ is equal to

- $\rho(x)$ if $\varphi = x$ and $x \in \Omega$
- $\varphi_1|_\rho \wedge \varphi|_\rho$ if $\varphi = (\varphi_1 \wedge \varphi_2)$
- ....

Exercise

Find the value of $((x_1 \vee x_2) \vee x_3) \wedge x_4$ with respect to $\rho = x_1 = T \quad x_2 = F \quad x_3 = F \quad x_4 = T$.

$x_1|_\rho = T$

$x_2|_\rho = F$

$x_1 \vee x_2|_\rho = T$

$(x_1 \vee x_2) \vee x_3|_\rho = T$

$x_4|_\rho = T$

$((x_1 \vee x_2) \vee x_3) \wedge x_4|_\rho = T$

Lemma

Let $\varphi_1, \varphi_2, \varphi_3$ be prop formulas on Ω and ρ be an assignement to Ω

Then

- $((\varphi_1 \wedge \varphi_2) \wedge \varphi_3)|_\rho = (\varphi_1 \wedge (\varphi_2 \wedge \varphi_3))|_\rho$
- $((\varphi_1 \vee \varphi_2) \vee \varphi_3)|_\rho = (\varphi_1 \vee (\varphi_2 \vee \varphi_3))|_\rho$
- $\neg(\varphi_1 \vee \varphi_2)|_\rho = (\neg\varphi_1 \wedge \neg\varphi_2)|_\rho$
- $\neg(\varphi_1 \wedge \varphi_2)|_\rho = (\neg\varphi_1 \vee \neg\varphi_2)|_\rho$
- $\neg\neg\varphi_1|_\rho = \varphi|_\rho$

Note that if $\Omega = \{x_1 \ldots x_n\}$
any formula φ defines a function
$\{T, F\}^n \to \{T, F\}$ (a Boolean function)

Theorem

For any Boolean function $f : \{T, F\}^n \to \{T, F\}$
there is a formula φ on $\{x_1 \ldots x_n\}$ s.t.

$$f(v_1 \ldots v_n) = \varphi\big|_{x_1 = v_1 \ldots x_n = v_n}$$

for any $v_1 \ldots v_n \in \{T, F\}$

x_1	x_2	x_3	
T	T	T	T
T	T	F	F
⋮			

bool f(bool, bool, bool)

x_1	x_2	
T	T	F
T	F	T
F	T	T
F	F	F

$$\boxed{\begin{array}{l} \text{if } x_1 == T \wedge x_2 == T \\ \quad \text{return } F \end{array}}$$

if $x_1 \wedge \neg x_2$ $\quad x_2 == F$
 return T

if $\neg x_1 \wedge x_2$
 return T

$$\boxed{\begin{array}{l} \text{if } \neg x_1 \wedge \neg x_2 \\ \quad \text{return } F \end{array}}$$

return F

?

$(x_1 \wedge x_2) \vee (\neg x_1 \wedge x_2)$

$x_1 == T$

x_1

if $(x_1 \wedge x_2) \vee (\neg x_1 \wedge x_2)$
 return T

Definition

Let $F_1 \dots F_n$ be p. formulas.

Then

- $\bigvee_{i=1}^{1} F_i = F_1 \qquad \bigwedge_{i=1}^{1} F_i = F_1$
- $\bigvee_{i=1}^{k+1} F_i = \left(\bigvee_{i=1}^{k} F_i\right) \vee F_{k+1}, \qquad \bigwedge_{i=1}^{k+1} F_i = \left(\bigwedge_{i=1}^{k} F_i\right) \wedge F_{k+1}$

Similarly if $S \subseteq [n]$ we can define

$$\bigvee_{i \in S} F_i \qquad \bigwedge_{i \in S} F_i$$

Consider $\varphi = \bigvee_{\substack{v \in \{T,F\}^n \\ \text{s.t. } f(v)=T}} \left(\bigwedge_{i=1}^{n} x_i^{v_i} \right),$

where $x_i^{u} = \begin{cases} x_i & \text{if } u = T \\ \neg x_i & \text{if } u = F \end{cases}$

$$\begin{array}{ccc} & v_1 & v_2 \\ v = (& T & F \,) \end{array}$$

$$v = (F \;\; T)$$

$$\bigwedge_{i=1}^{1} x_i^{v_i} = x_1^{v_1} = x_1$$

$$\bigwedge_{i=1}^{2} x_i^{v_i} = x_1 \wedge x_2^{v_2} = x_1 \wedge \neg x_2$$

$$\bigwedge_{i=1}^{1} x_i^{v_i} = x_1^{v_1} = \neg x_1$$

$$\bigwedge_{i=1}^{2} x_i^{v_i} = \neg x_1 \wedge x_2^{v_2} = \neg x_1 \wedge x_2$$