Name: ____________________

Pid: ____________________

**Note that since this class is about proofs, every statement in the final exam should be proved. The only exceptions are statements that were proven in previous homework or midterms and statements proven earlier in the class.**

1. (20 points) Check all the correct statements.
   - ◯ For any positive integer $n$, $|[n]^2| > |[n]|$.
   - ◯ If you have 15 balls in 5 boxes, then there is a box with at least 3 balls.
   - ◯ There are 12 elements in the set $[6] \cup \{x \ : \ 12 < x < 19\}$.
   - ◯ There are 10 ways to select 2 objects out of 3.
   - ◯ The set $[2]^{[3]}$ has 8 elements.
   - ◯ The function $f : [2] \to [3]$ such that $f(x) = x$ belongs to the set $[2]^{[3]}$.
   - ◯ The function $-x$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.
   - ◯ $p \wedge \neg p$ is always true.
   - ◯ There is an injection from $[5] \times [5]$ to $[25]$.
   - ◯ A function $f : X \to Y$ is an injection iff the set $\{x \in X \ : \ f(x) = y\}$ has cardinality at most 1 for all $y \in Y$.

**Solution:**

1. Note that $|[n]^2| = n^2$ and $|[n]| = n$. It is easy to see that $n^2 > n$ for $n > 1$ and $n^2 = n$ for $n = 1$.
2. By the generalized pigeonhole principle, there is a box with at least $\frac{15}{5} = 3$ balls.
3. Note that these sets are disjoint, thus the size of the union is equal to $6 + |\{x \ : \ 12 < x < 19\}| = 6 + 6 = 12$.
4. There are $\frac{3 \cdot 2}{2} = 3$ ways to select 2 objects out of 3.
5. We proved that the set $Y^X$ has $|Y|^{|X|}$ elements, so the set $[2]^{[3]}$ has 8 elements.
6. Note that the set $[2]^{[3]}$ is the set of functions from $[3]$ to $[2]$, thus $f$ does not belong to this set.
7. It is clearly a bijection, since it is the inverse of itself.
8. It is not always true, since false and not false is false.
9. Yes, it is true, moreover, there is a bijection.
10. Yes, it is true, it is exactly the definition of the injection, written a bit differently.

2. (10 points) Prove the following recurrent formula:

$$S(n,k) = k \cdot S(n-1,k-1) + k \cdot S(n-1,k),$$

where $S(n,k)$ denotes the number of surjective functions from $[n]$ to $[k]$.

**Solution:** Note that there are two possible situations when we define a surjection $f : [n] \to [k]$: either $f(i) \neq f(n)$ for any $i < n$ or $f(i) = f(n)$ for some $i < n$.

- Assume, we need to construct a surjection $f : [n] \to [k]$ such that $f(i) \neq f(n)$. In this case we have $k$ ways to select $f(n)$ and $S(n-1,k-1)$ ways to select $f|_{[n-1]}$; i.e. there are $kS(n-1,k-1)$ such surjections $f$.
- Assume, we need to construct a surjection $f : [n] \to [k]$ such that $f(i) \neq f(n)$. In this case we have $k$ ways to select $f(n)$ and $S(n-1,k)$ ways to select $f|_{[n-1]}$; i.e. there are $kS(n-1,k)$ such surjections $f$.

Thus by the additive principle, the number of surjections from $[n]$ to $[k]$ is equal to $k \cdot S(n-1,k-1) + k \cdot S(n-1,k)$.

3. (10 points) Show that $\sum\limits_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for any positive integer $n \geq 2$.

**Solution:** Imagine that we have $n$ people in a group and we need to choose a subgroup of them and a head of this subgroup.

It is easy to see that we have $n$ ways to select the head and $2^{n-1}$ ways to select the rest of the subgroup. On the other hand, if we know that the subgroup has $i$ members, then there are $\binom{n}{i}$ ways to select the subgroup and $i$ ways to select the head, thus there are $\sum_{k=1}^{n} k\binom{n}{k}$ ways to select the subgroup and its leader. As a result, $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$.

4. (10 points) In school there are three clubs and for any two students there is at least one club such that both of them are in this club. Show that for some club 2/3 of the students are in this club.

**Solution:** We need to consider the following two cases.

1. If there is a student participating in only one club. In this case all the other students are in this club as well. Thus more than 2/3 students in this club.
2. Otherwise, all the students participate in at least 2 clubs. Let $p_1$, $p_2$, and $p_3$ be the numbers of members in these clubs. Note that $p_1 + p_2 + p_3 \geq 2n$. Thus $p_i \geq 2/3n$ for some $i \in \{1, 2, 3\}$.

5. (10 points) We say that a function $f : \{0,1\}^n \to \{0,1\}$ depends on the $i$th argument iff for some $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in \{0,1\}$

$$f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n).$$

We also say that the function $f$ depends on all the arguments iff for all $i \in [n]$ it depends on $i$th argument. Find the number of functions $f : \{0,1\}^n \to \{0,1\}$ depending on all arguments.

**Solution:** Let $S \subseteq [n]$ and $F_i$ be the set of function from $\{0,1\}^n$ to $\{0,1\}$ not depending on the $i$th input. It is easy to see that there are $2^{2^{n-1}}$ elements in $F_i$ and moreover, there are $2^{2^{n-|S|}}$ elements in the set $\cap_{i \in S} F_i$. Thus by the inclusion-exclusion principle, there are

$$\sum_{S \subseteq [n] \,:\, S \neq \emptyset} (-1)^{|S|+1} 2^{2^{n-|S|}} = \sum_{k=1}^{n} (-1)^k 2^{2^{n-k}} \binom{n}{k}$$

functions not depending on at least one argument. As a result, the answe is $2^{2^n} - \sum_{k=1}^{n} (-1)^k 2^{2^{n-k}} \binom{n}{k}$.

6. (10 points) How many integer numbers from 0 to 999 are having at least one digit equal to 7.

**Solution:** First note that any number from 0 to 999 may be expressed using 3 digits (we allow to use leading 0s to express the numbers like 1). Thus we need to find the size of the set

$$\{(a_1, a_2, a_3) \ : \ a_1, a_2, a_3 \in \{0, 1 \ldots, 9\} \text{ and } 7 \in \{a_1, a_2, a_3\}\}.$$

Let $S \subseteq [3]$, it is easy to see that $\bigcap_{i \in S} \{(a_1, a_2, a_3) \ : \ a_1, a_2, a_3 \in \{0, 1 \ldots, 9\} \text{ and } a_i = 7\}\}$ has $10^{|S|}$ elements. As a result, the answer is $1000 - 3 \cdot 10^2 + 3 \cdot 10 - 1$.

7. (10 points) Let us consider Young's geometry, it is a theory with undefined terms: point, line, is on, and axioms:

   1. there exists at least one line,
   2. every line has exactly three points on it,
   3. not all points are on the same line,
   4. for two distinct points, there exists exactly one line on both of them,
   5. if a point does not lie on a given line, then there exists exactly one line on that point that does not intersect the given line.

   Show that for every point, there are exactly four lines on that point.

**Solution:** First, we prove that for every point, there is a line not on that point. Let $p$ be some point. By the first axiom, there is a line $\ell$. Assume that $p$ is on $\ell$ (otherwise we proved the statement). By axiom 2, there are two other points $p_1$ and $p_2$ on this line. By axiom 3, there is a point $q$ not on $\ell$. Finally, by axiom 4, there is a line $\ell'$ on $p_1$ and $q$. Note that $\ell' \neq \ell$, thus $p$ is not on $\ell'$.

Now we are ready to prove that there are at least four lines. Let $p$ be a point and $\ell$ be a line such that $p$ is not on $\ell$. By axiom 2, there are three points $p_1$, $p_2$, and $p_3$ on $\ell$. By axiom 4, there are lines $\ell_1$, $\ell_2$, and $\ell_3$ such that $p$ is on all of them and $p_i$ is on $\ell_i$ for $i \in [3]$. By axiom 5, there is a line $\ell_4$ containg $p$ but $p_1$, $p_2$, and $p_3$ are not on $\ell_4$. Thus, there are at least four lines through $p$.

Let us now prove that there is no other line $\ell_5$ such that $p$ is on $\ell_5$. By axiom 5, there is a point $p_i$ such that $p_i$ is on both $\ell_5$ and $\ell$. But this contradicts to axiom 4, since $p_i$ is also on $\ell_i$.