Chapter 5

Sets

5.1 The Intuitive Definition of a Set

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Sets

A set is one of the two most important concepts in mathematics. Many mathematical statements involve “an integer n” or “a real number a”. Set theory notation provides a simple way to express that a is a real number. However, this language is much more expressible and it is impossible to imagine modern mathematics without this notation.

As in the previous chapter it is difficult to define a set formally so we give a less formal definition which should be enough to use the notation. A *set* is a well-defined collection of objects. Important examples of sets are:

1. $\mathbb{R}$ a set of reals,
2. $\mathbb{Z}$ the set of integers[1],
3. $\mathbb{N}$ the set of natural numbers[2],
4. $\mathbb{Q}$ a set of rational numbers,
5. $\mathbb{C}$ a set of complex numbers.

Usually, sets are denoted by single letter.

Objects in a set are called *elements* of the set and we denote the statement “x is in the set E” by the formula $x \in E$ and the negation of this statement by

[1]“Z” stands for the German word Zahlen (“numbers”).

[2]Note that in the literature there are two diffirent traditions: in one 0 is a natural number, in another it is not; in this book we are going to assume that 0 is not a natural number.

$x \notin E$. For example, we proved that $\sqrt{2} \notin \mathbb{Q}$[3].

Exercise 5.1. *Which of the following sets are included in which? Recall that a number is prime iff it is an integer greater than* 1 *and divisible only by* 1 *and itself.*

1. *The set of all positive integers less than* 10.
2. *The set of all prime numbers less than* 11.
3. *The set of all odd numbers greater than* 1 *and less than* 6.
4. *The set of all positive integers less than* 10.
5. *The set whose only elements are* 1 *and* 2.
6. *The set whose only element is* 1.
7. *The set of all prime numbers less than* 11.

5.2 Basic Relations Between Sets

Many problems in mathematics are problems of determining whether two description of sets are describing the same set or not. For example, when we learn how to solve quadratic equations of the form $ax^2 + bx + c = 0$ $(a, b, c \in \mathbb{R})$ we learn how to list the elements of the set $\{x \in \mathbb{R} \, : \, ax^2 + bx + c = 0\}$.

We say that two sets A and B are equal if they contain the same elements (we denote it by $A = B$). If all the elements of A belong to B we say that A is a subset of B and denote it by $A \subseteq B$[4].

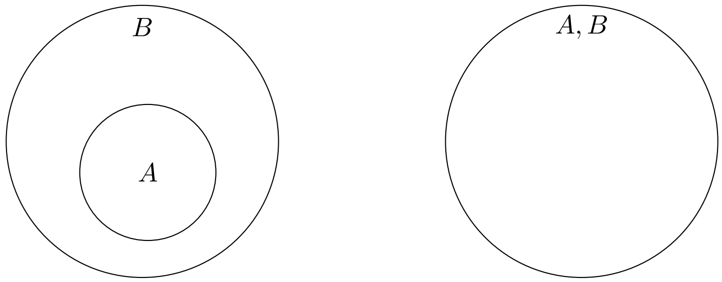
For example, $\mathbb{Q} \subseteq \mathbb{R}$ since any rational number is also a real number. A special set is an empty set i.e. the set that does not have elements, we denote it $\emptyset$.

5.2.1 Diagrams

If we think of a set A as represented by all the points within a circle or any other closed figure, then it is easy to represent the notion of A being a subset of another set B also represented by all the points within a circle. We just put a circle labeled by A inside of the circle labeled by B. We can also diagram an equality by drawing a circle labeled by both A and B. (see fig. 5.1). Such diagrams are called Euler diagrams and it is clear that one may draw Euler diagrams for more than two sets.

[3] The symbol $\in$ was first used by Giuseppe Peano 1889 in his work "Arithmetices principia, nova methodo exposita". Here he wrote on page X: "The symbol $\in$ means is. So $a \in b$ is read as a *is* a b; ..." The symbol itself is a stylized lowercase Greek letter epsilon ("ϵ"), the first letter of the word εστι, which means "is".

[4] In the literature there are three symbols for "subset": $\subseteq$, $\subsetneq$, and $\subseteq$. $A \subseteq B$ means that A is a subset of B and we allow $A = B$ and $A \subsetneq B$ means that A is a subset of B and we forbid $A = B$. However, there is a problem with the third symbol, some people use it as a synonym of $\subseteq$ and some use it as a synonym of $\subsetneq$. Due to this ambiguity we are going to avoid using it in this book.



(a) $A \subseteq B$ (b) $A = B$

Figure 5.1: Euler diagrams

5.2.2 Descriptions of Sets

Listing elements. There are several ways to construct a set, the simplest one is just to list them. For example

1. $\{1, 2, \pi\}$ is the set consisting of three elements 1, 2, and π, and
2. $\{1, 2, 3, \ldots\}$ is the set of all positive integers i.e. it is the set $\mathbb{N}$.

Conditional definitions. We may also describe a set using some constraint e.g we may list all the even numbers using the following formula $\{n \in \mathbb{Z} \ : \ n \text{ is even}\}$ (we read it as "the set of all integers n such that n is even").

Using this we may also define the set of all integers from 1 to m, we denote it $[m]$; i.e. $[m] = \{n \in \mathbb{N} \ : \ 0 < n \leq m\}$.

Constructive definitions. Another way to construct a set of all even numbers is to use the constructive definition of a set: $\{2k \ : \ k \in \mathbb{Z}\}$.

We may also describe a set of rational numbers using this description: $\mathbb{Q} = \{a/b \ : \ a \in \mathbb{Z}, b \in \mathbb{N}\}$ (note that we may also use a mix of a conditional and constructive definitions, $\mathbb{Q} = \{a/b \ : \ a, b \in \mathbb{Z}, b \neq 0\}$).

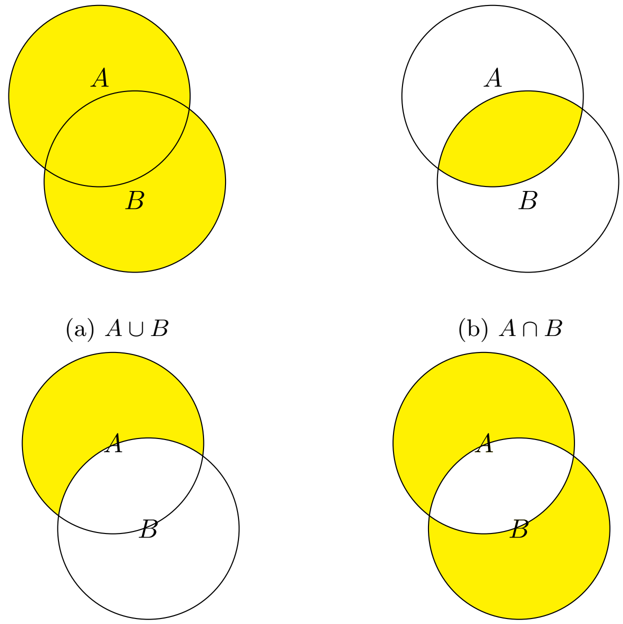
Exercise 5.2. *Describe a set of perfect squares using constructive type of definition.*

5.2.3 Disjoint Sets

Two sets are *disjoint* iff they do not have common elements. We also say that two sets are *overlapping* iff they are not disjoint i.e. they share an element.

More generally, $A_1, \ldots, A_\ell$ are pairwise disjoint iff A_i is disjoint with A_j for all $i \neq j \in [\ell]$

Exercise 5.3. *Of the sets in Exercise 5.1, which are disjoint from which?*



(a) $A \cup B$ (b) $A \cap B$

(c) $A \setminus B$ (d) $A \Delta B$

Figure 5.2: Operations over the sets

5.3 Operations over Sets.

Another way to describe a set is to apply operation to other sets. Let A and B be sets.

The first example of the operations on sets is the *union* operation. The union of A and B is the set containing all the elements of A and all the elements of B i.e. $A \cup B = \{x \ : \ x \in A \text{ or } x \in B\}$[5].

Another example of such an operation is *intersection*. The intersection of A and B is the set of all the elements belonging to both A and B i.e $A \cap B = \{x \ : \ x \in A \text{ and } x \in B\}$[6].

The third operation we are going to discuss this lecture is *set difference*. If A and B are some sets, then $A \setminus B = \{x \ : \ x \in A \text{ and } x \notin B\}$.

The last operation is *symmetric difference*. If A and B are some sets, then $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Note that alternatively $A\Delta B = (A \cup B) \setminus (A \cap B)$

Exercise 5.4. *Describe the set* $\{n \in \mathbb{N} \ : \ n \textit{ is even}\} \cap \{3n \ : \ n \in \mathbb{N}\}$.

[5] Note that this definition is not correct since in the conditional definitions we have to specify the set x belongs to and we cannot do this here.

[6] You may notice that in the definition of the union we use disjunction and in the definition of intersection we use conjunction. Actually this is a the reason the symbol of the conjunction is similar to the symbol of intersection and the symbol of the disjunction is similar to the symbol of union.

Theorem 5.1. *Let A, B, and C be some sets. Then we have the following identities.*

associativity: $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$.

commutativity: $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$.

distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. One may prove these properties using the Euler diagrams. Alternatively they can be proven by definitions. Let us prove only the first part of the distributivity, the rest is Exercise 5.5.

Our proof consists of two parts in the first part we prove that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Suppose that $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in (B \cap C)$.

- If $x \in A$, then $x \in (A \cup B)$ and $x \in (A \cup C)$ i.e. $x \in ((A \cup B) \cap (A \cup C))$.
- If $x \in (B \cap C)$, then $x \in B$ and $x \in C$. Which implies that $x \in (A \cup B)$ and $x \in (A \cup C)$. As a result, $x \in ((A \cup B) \cap (A \cup C))$.

$\square$

Exercise 5.5. *Prove the rest of the equalities in Theorem 5.1.*

Probably the most difficult concept connected to sets is the concept of a power set. Let A be some set, then the set of all possible subsets of A is denoted by 2^A (sometimes this set is denoted by $\mathcal{P}(A)$) and called the power set of A. In other words $2^A = \{B \ : \ B \subseteq A\}$.

Warning: Please do not forget about two extremal elements of the power set 2^A: the empty set and A itself.

For example if $A = \{1, 2, 3\}$, then

$$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} .$$

5.4 The Well-ordering Principle

Using the set notation we may finally justify the proof of the statement that $2^n > n$ for all positive integers n from the video about mathematical induction. In order to do this let us first formulate the following theorem.

Theorem 5.2. *Let $A \subseteq \mathbb{Z}$ be a non-empty set. We say that $b \in \mathbb{Z}$ is a lower bound for the set A iff $b \leq a$ for all $a \in A$. Additionally, we say that the set A is bounded if there is a lower bound for A.*

Given this, if A is bounded, then there is a lower bound $a \in A$ for the set A (we say that a is the minimum of the set A).

Note that this theorem also states that any subset of natural numbers have a minimum.

Recall that we wish to prove that $2^n > n$ for all positive n. Assume that it is not true, in this case the set $A = \{n \in \mathbb{N} \ : \ 2^n < n\}$ is non-empty. Denote by n_0 the minimum of the set A, n_0 exists by Theorem 5.2. We may consider the following two cases.

- If $n_0 = 1$, then it leads to a contradiction since $2 = 2^1 > 1$.
- Otherwise, note that $1 \leq n_0 - 1 < n_0$, hence, $2^{n_0 - 1} > n_0 - 1$. So $2^{n_0} > 2n_0 - 2 \geq n_0$. Which is a contradiction with the definition of n_0.

Finally, we prove Theorem 5.2.

Proof of Theorem 5.2. Let b be a lower bound for the set A. Assume that there is no minimum of the set A. Let $P(n)$ be the statement that $n \notin A$.

First, we are going to prove that $P(n)$ is true for all $n \geq b$. The base case is true since if $b \in A$, then b is the minimum of A which contradicts to the assumption that there is no minimum of A. The induction step is also clear, by the induction hypothesis we know that $P(b)$, $\dots$, $P(k)$ are true, hence, $(k+1) \in A$ implies that $k+1$ is the minimum of A.

Now we prove that A is empty. Assume the opposite i.e. assume that there is $x \in A$. Note that $x \geq b$ since b is a lower bound of A. However, $P(x)$ is true which implies that $x \notin A$. Therefore the assumption was false and A is empty, but this contradicts to the fact that A is non- empty. $\square$

End of The Chapter Exercises

5.6 Find the power sets of $\emptyset$, $\{1\}$, $\{1,2\}$, $\{1,2,3,4\}$. How many elements in each of this sets?

5.7 Prove that

- $A \subseteq B \iff A \cup B = B$,
- $A \subseteq B \iff A \cap B = A$.

5.8 Let A be a subset of a set U we call this set a universe. We say that the set $\overline{A} = U \setminus A$ is a complement of A in U. Show the following equalities

- $\overline{\overline{A}} = A$.
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

5.9 Let us define an intersection of more than two sets as follows. Let $A_1, \dots,$ A_n be some sets. Then

- $\bigcap_{i=1}^{1} A_i = A_1$ and

- $\bigcap_{i=1}^{k+1} A_i = \left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}$.

Show that $\bigcap_{i=1}^{n} \{x \in \mathbb{N} \ : \ i \leq x \leq n\} = \{n\}$ for all integers $n > 0$.

5.10 Let us define a union of more than two sets as follows. Let $A_1, \ldots, A_n$ be some sets. Then

- $\bigcup_{i=1}^{1} A_i = A_1$ and
- $\bigcup_{i=1}^{k+1} A_i = \left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}$.

Show that $\bigcup_{i=1}^{n} [i] = [n]$ for all integers $n > 0$.

5.11 Let Ω be some set and $A_1, \ldots, A_n \subseteq \Omega$. Show that $\bigcup_{i=1}^{n} A_i = \{x \in \Omega \ : \ \exists i \in [n] \ x \in A_i\}$.

5.12 Let $A_1, \ldots, A_n$ be some sets. Show that $\bigcup\limits_{i=1}^{n} (A_i \cap B) = (\bigcup_{i=1}^{n} A_i) \cap B$.

5.13 Show that $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.