Name: ______________________

Pid: ______________________

1. (50 points) Check all the correct statements (in this question only the answers will be graded).

   ○ $\gcd(24, 18) = 6$.

   ○ The function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ such that $f(x) = \arctan x$ is a bijection.

   ○ The cardinality of the set $F(X, [3]) = (4^n)^3$, where $X = F([4], [n])$.

   ○ The cardinality of the set $I([3], [n]) = n(n-1)(n-2)$.

   ○ $\binom{10}{2} = 90$.

Solution:

1. Note that $D(24) = \{1, 2, 3, 4, 6, 8, 12, 24\}$ and $D(18) = \{1, 2, 3, 6, 9, 18\}$. Hence, $\gcd(24, 18) = 6$.
2. No it is not a bijection since arctan is increasing function, hence, the value of $\mathrm{Im} f \subseteq [f(-\frac{\pi}{2}), f(\frac{\pi}{2})]$.
3. The cardianality of the set $X = F([4], [n])$ is equal to n^4, hence, the cardinality of the set $F(X, [3])$ is equal to 3^{n^4}.
4. $\binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = 45$.

2. (a) (5 points) Let n, a, and b be some integers. Show that if two numbers a and b have the same reminders when divided by n, then $a-b$ is divisible by n.

Solution: There are integers k, ℓ and r such that $a = kn + r$ and $a = \ell n + r$ since a and b have the same reminder when divided by n.
Note that $a - b = (k - \ell)n$, hence, is divisible by n.

(b) (5 points) Prove that for every integers $a_1, \dots, a_n$ there are $k > 0$ and $\ell \geq 0$ such that $k + \ell \leq n$ and $\sum_{i=k}^{k+\ell} a_i$ is divisible by n.

Solution: Let us consider the function $f : \{0, 1, \dots, n\} \to \{0, 1, \dots, n-1\}$ such that $f(i)$ is equal to the reminder of $\sum_{j=1}^{i} a_j$ (if $i < 1$, the sum is equal to 0) when divided by n. By the pigeonhole prinnciple there are $i_0 < i_1$ such that $f(i_0) = f(i_1)$; hence, $f(i_1) - f(i_0) = \sum_{j=1}^{i_1} a_j - \sum_{j=1}^{i_0} a_j = \sum_{j=i_0+1}^{i_1} a_j$ is divisible by n.

3. (10 points) We say that sets A_1, A_2, and A_3 are pairwise disjoint iff $A_i \cap A_j = \emptyset$ for every $i \neq j \in [3]$. Construct a bijection from $\{0,1,2,3\}^n$ to $\{(A,B,C) \mid A,B,C \subseteq [n] \text{ and } A,B,C \text{ are pairwise disjoint}\}$

Solution: Let us consider the function $f : \{0,1,2,3\}^n \to \{(A,B,C) \mid A,B,C \subseteq [n] \text{ and } A,B,C \text{ are pairwise disjoint}\}$ such that $f(x_1,\ldots,x_n) = (A_x,B_x,C_x)$, where $A_x = \{i \in [n] \mid x_i = 1\}$, $B_x = \{i \in [n] \mid x_i = 2\}$ $C_x = \{i \in [n] \mid x_i = 3\}$.

It is easy to see that the function is a bijection since we may define the inverse of this function $e : \{(A,B,C) \mid A,B,C \subseteq [n] \text{ and } A,B,C \text{ are pairwise disjoint}\} \to \{0,1,2,3\}^n$ such that $e(A,B,C) = (x_1,\ldots,x_n)$, where $x_i = \begin{cases} 1 & \text{if } i \in A \\ 2 & \text{if } i \in B \\ 3 & \text{if } i \in C \\ 0 & \text{otherwise} \end{cases}$.

- Let $f(e(A,B,C)) = (A',B',C')$ and $e(A,B,C) = (x_1,\ldots,x_n)$. Note that $x_i = 1$ iff $i \in A$ and $i \in A'$ iff $x_i = 1$; hence $i \in A$ iff $i \in A'$. In other words, $A = A'$. Similarly we may consider other cases (we use the fact that A, B, and C to show that constraints in the definition of e cannot be satisfied simultaneosly).
- Let $e(f(x_1,\ldots,x_n)) = (x'_1,\ldots,x'_n)$ and $f(x_1,\ldots,x_n) = (A,B,C)$. Note that $i \in A$ iff $x_i = 1$ and $x'_i = 1$ iff $i \in A$; hence $x_i = 1$ iff $x'_i = 1$. Similarly we may prove for 0, 2, and 3 and as a result, we proved that $x_i = x'_i$.

4. (10 points) How many numbers from [999] are not divisible neither by 3, nor by 5, nor by 7.

Solution: Let $S_n = \{i \in [999] \mid i \text{is divisible by n}\}$. Note that $S_3 \cap S_5 = S_{15}$, $S_3 \cap S_7 = S_{21}$, $S_5 \cap S_7 = S_{35}$, and finally, $S_3 \cap S_5 \cap S_7 = S_{105}$. Additinally, $|S_3| = 999/3 = 333$, $|S_5| = \lfloor 999/5 \rfloor = 199$, $|S_7| = \lfloor 999/7 \rfloor = 142$, $|S_1 5| = \lfloor 999/15 \rfloor = 66$, $|S_{21}| = \lfloor 999/21 \rfloor = 47$, $|S_{35}| = \lfloor 999/35 \rfloor = 28$, and $|S_{105}| = \lfloor 999/105 \rfloor = 9$. As a result, by the inclusion-exclusion principle, the answer is $999 - 333 - 199 - 142 + 66 + 47 + 28 - 9 = 457$.

5. (10 points) Let m be some integer. Show that product of m consecutive integers is divisible by $m!$.

Solution: In other words we need to show that for any integer n, $\frac{n\cdot(n+1)\cdot\dots\cdot(n+m-1)}{m!}$ is an integer. But one may notice that $\frac{n\cdot(n+1)\cdot\dots\cdot(n+m-1)}{m!} = \binom{n+m-1}{m}$ which is an integer.