Name: ____________________

Pid: ____________________

Note that since this class is about proofs, every statement in the midterm should be proved. The only exceptions are statements that were proven in previous homework or midterms and statements proven earlier in the class.

1. (90 points) Check all the correct statements (in this question only the answers will be graded). **Please check only the correct statements, this question is graded automatically.**

   ◯ The stements $\neg(p \wedge (\neg p \vee q)) \vee q$ is always true.

   ◯ The sets $\{k^2 \mid k \in \mathbb{N}\}$ and $\{k^2 \mid k \in \mathbb{Z}\}$ are equal.

   ◯ The sets $\{2k \mid k \in \mathbb{Z}\} \setminus \{3k \mid k \in \mathbb{Z}\}$ and $\{6k+2, 6k+4 \mid k \in \mathbb{Z}\}$ are equal.

   ◯ The sets $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q}$ are equipotent.

   ◯ The set $[n]^2 \setminus \{(i,j) \mid i \neq j\}$ has cardinality $n^2 - n$.

   ◯ The cardinality of the set $F([3], X)$ is equal to 4^{3n}, where $X = F([4], [n])$.

   ◯ The cardinality of the set $\{A \subseteq [n] \mid |A| = n-1\}$ is equal to n.

   ◯ $\binom{3}{2} = 3$.

   ◯ $\gcd(6, 4) = 2$.

2. (10 points) Show that $F(\mathbb{N}, \{0,1\})$ is equipotent to $F(\mathbb{N}, \{0,1,2\})$

3. (10 points) Let $\oplus$ be a Boolean operation such that for any $a, b \in \{0,1\}$, $a \oplus b = 1$ iff $a \neq b$. A function $f : \{0,1\}^n \to \{0,1\}$ can be represented as a monomial iff $f(x_1, \ldots, x_n) = x_{i_1} \cdot x_{i_2} \cdot \cdots \cdot x_{i_k}$ for all $x_1, \ldots, x_n \in \{0,1\}$, where $k \geq 0$ and $i_1, \ldots, i_k \in [n]$. Finally we say that a function $f : \{0,1\}^n \to \{0,1\}$ can be represented as a Zhegalkin polynomial iff $f(x_1, \ldots, x_n) = g_1(x_1, \ldots, x_n) \oplus \cdots \oplus g_\ell(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in \{0,1\}$, where $g_1, \ldots, g_\ell : \{0,1\}^n \to \{0,1\}$ are different functions representable as monomials.

   Show that any function $f : \{0,1\}^n \to \{0,1\}$ is representable as a Zhegalkin polynomial.

4. Let us consider the following theory, it is a theory with undefined terms: element, plus, and times (if a and b are elements, we denote a times b by $a \cdot b$ and a plus b by $a + b$), and axioms:

   Associativity: for all elements a, b, and c, $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

   Commutativity: for all elements a and b, $a + b = b + a$ and $a \cdot b = b \cdot a$.

   Identity elements: there exist two different elements 0 and 1 such that $a + 0 = a$ and $a \cdot 1 = a$ for every element a.

   Inverses: for every element a, there exists an element, denoted $-a$, called the additive inverse of a, such that $a + (-a) = 0$ and moreover, for every $a \neq 0$, there exists an element, denoted by a^{-1} or $\frac{1}{a}$, called the multiplicative inverse of a, such that $a \cdot a^{-1} = 1$.

   Distributivity: for all elements a, b, and c, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

   Let $k \in \mathbb{N}$ and a be an element. Then we denote $a + a + \cdots + a$ (k times) by $k \cdot a$.

   We say that $p_a \in \mathbb{N}$ is a characteristic of an element a if $p_a \cdot a = 0$ but $q \cdot a \neq 0$ for all $q < p_a$.

   (a) (5 points) Show that if a and b are nonzero elements and p_a exists, then their characteristics are euqal.

   (b) (5 points) Show that if a is a nonzero element, then p_a is prime.

5. (10 points) Show that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

6. (10 points) Let $f_0 = 1$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{N}$. Show that $f_n \geq \left(\frac{3}{2}\right)^{n-2}$.

7. (10 points) Sasha is training for a triathlon. Over a 30 day period, he pledges to train at least once per day, and 45 times in all. Then there will be a period of consecutive days where he trains exactly 14 times.