# Hard satisfiable formulas for splittings by linear combinations

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# DPLL algorithms and unsatisfiable formulas

Let us run some DPLL algorithm on

$$(x \vee y) \wedge (\neg x \vee \neg y) \wedge (\neg y \vee t) \wedge (y \vee \neg t) \wedge (x \vee t) \wedge (\neg x \vee \neg t).$$

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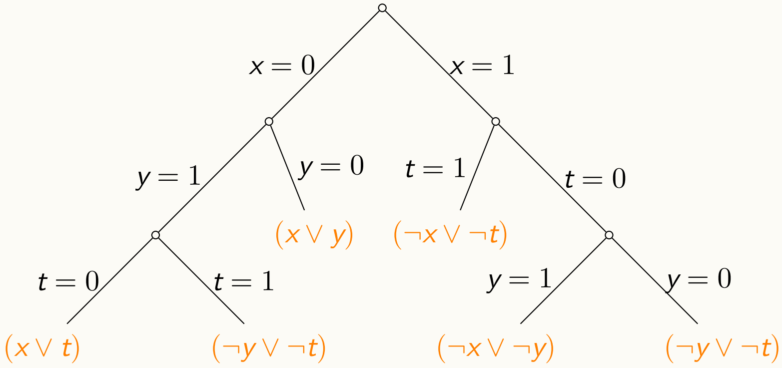

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*Tseitin formulas are exponentially hard for* DPLL, *i.e., unsatisfiable systems of linear equations are hard for* DPLL.

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**THEOREM (ALEKHNOVICH, HIRSCH, AND ITSYKSON, 2005)**

*Satisfiable linear systems are hard for myopic and drunken* DPLL *algorithms.*

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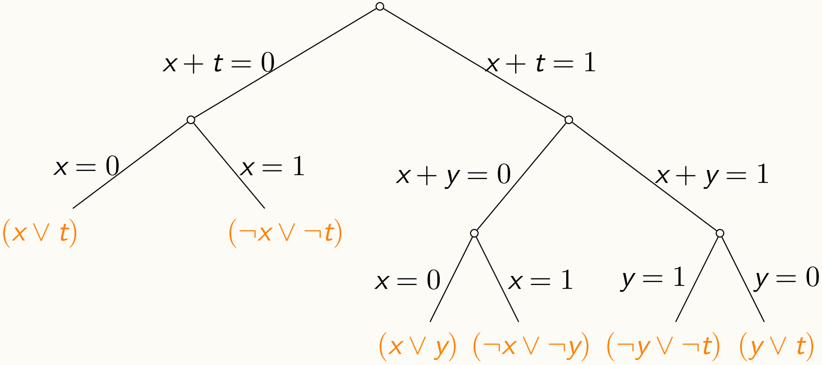

# Complexity of unsatisfiable formulas

| Formula | DPLL | Res | DPLL($\oplus$) |
|---|---|---|---|
| $\mathbb{F}_2$-linear systems | hard | hard | easy<br>[Itsykson and Sokolov 2014] |
| Perfect matching in $K_{2n+1}$ | $2^{\Theta(n \log n)}$ | $2^{\Theta(n)}$ | poly(n)<br>[Itsykson and Sokolov 2014] |
| $\mathrm{PHP}^n_{n+1}$ | $2^{\Theta(n \log n)}$ | $2^{\Theta(n)}$ | $2^{\Theta(n)}$<br>[Itsykson and Sokolov 2014]<br>[Oparin 2016] |
| $\mathrm{TS}^{\wedge}_{G,c}$ | $2^{\Theta(n)}$ | $2^{\Theta(n)}$ | $2^{\Omega(n^{\epsilon})}$<br>[Itsykson and Sokolov 2014] |
| Random 3-CNF | $2^{\Theta(n)}$ | $2^{\Theta(n)}$ | $2^{\Theta(n)}$<br>[Garlik and Kolodziejczyk 2017] |
| Lifted Pebbling | $2^{\Omega(n/\log n)}$ | $poly(n)$ | $2^{\Omega(n/\log n)}$<br>[Itsykson and Sokolov 2017] |

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**THEOREM**

*There exists an explicit family of satisfiable CNF formulas $\Psi_n$ such that any drunken* DPLL($\oplus$) *runs on $\Psi_n$ at least $2^{\Omega(n)}$ steps with probability at least $1 - 2^{-\Omega(n)}$.*

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## PLAN OF THE PROOF

- $\Psi_n$ is $\mathtt{PHP}^n_{n+1}$ plus one satisfying assignment;
- Prove that w.h.p. a drunken DPLL will make an incorrect substitution;
- Adopt the lower bound technique for $\mathtt{PHP}^n_{n+1}$.

# The Prover-delayer Game

Let $\varphi(x_1, x_2, \ldots, x_n)$ be a CNF formula; Prover and Delayer are playing the following game.

- Prover asks for the value of $x_i$ for some $i \in [n]$;
- Delayer gives an answer from $\{0, 1\}$ or "Choose any"; In the case of "Choose any" Prover chooses the value from $\{0, 1\}$
- Delayer earns $1$ coin for every answer "Choose any";
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**THEOREM (CF. PUDLAK AND IMPAGLIAZZO, 2001)**

*If there is a strategy for Delayer such that for every Prover's strategy, Delayer earns at least $t$ coins, then the size of any decision tree for $\varphi$ is at least $2^t$.*

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- The formulas has $n(n+1)$ variables $\mathfrak{P}_n = \{p_{i,j} \ : \ i \in [n+1], j \in [n]\}$ (informally, $p_{i,j} = 1$ iff the $i$th pigeon is in the $j$th hole).

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- We say that a substitution $\pi$ is proper if it satisfies all the short clauses.
- $\pi$ properly implies $\rho$ if any proper assignment that satisfy $\pi$ also satisfies $\rho$;
- A proper rank of a substitution is the minimal number of equalities that properly imply all the other equalities.

# A Lower Bound on Unsatisfiable Instances

**LEMMA**

*Let a substitution $\pi$ to the variables $\mathfrak{P}_n$ has a proper rank at most $n-1$ and can be extended to a proper substitution. Then for all $i \in [n+1]$ there is a proper solution that satisfies $p_{i,1} \vee p_{i,2} \vee \cdots \vee p_{i,n}$.*

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**PROOF.**

Strategy of Delayer is the following: if the value of $x_i$ is properly implied from the current substitution, then return it, otherwise, return "Choose any".

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Hence, Delayer earns at least $\frac{n-1}{2}$ coins. □

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Consider the moment when the solution is found, the current substitution has proper rank at least $n - 1$. Consider the moments on the acceptance branch when the proper rank grows $0 \to 1$, $1 \to 2$, ..., $\frac{n-1}{2} - 1 \to \frac{n-1}{2}$.

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Consider the moment when the solution is found, the current substitution has proper rank at least $n-1$. Consider the moments on the acceptance branch when the proper rank grows $0 \to 1$, $1 \to 2$, ..., $\frac{n-1}{2} - 1 \to \frac{n-1}{2}$.
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